Tutorial 2 (Jan 22, 24)

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Q1) Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

Sol) First attempt : by direct computation

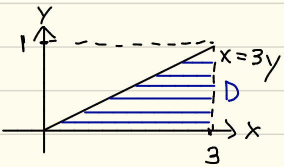
$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \boxed{?}$$

Correct attempt : Use Fubini's Theorem

Step 1 : Write the region of integration D

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \,;\, 3y \leq x \leq 3\}$$

Picture :



Step 2 : Describe D using different order of variables :

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3 \,;\, 0 \leq y \leq \tfrac{x}{3}\}$$

Step 3 : Reverse the order of integration by Fubini's Thm

$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\, dy\, dx = \int_0^3 \left[y e^{x^2} \right]_0^{\frac{x}{3}} dx$$

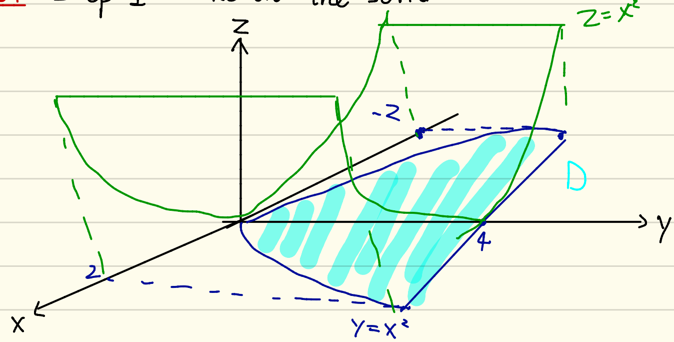
$$= \frac{1}{3} \int_0^3 x e^{x^2}\, dx$$

$$= \frac{1}{6} \left[e^{x^2} \right]_0^3 = \frac{1}{6}(e^9 - 1)$$

Q2) Find the volume of the solid bounded by the cylinders

$z=x^2$, $y=x^2$ and the planes $z=0$, $y=4$.

Sol Step 1: Sketch the solid



Step 2: Write the region of integration

$$D=\{(x,y)\in\mathbb{R}^2 \mid -2\leq x\leq 2\,;\ x^2\leq y\leq 4\}$$

Step 3: Compute the volume by a double integral

$$\text{Volume} = \iint_D f(x,y)\,dA\,, \quad \text{where } f(x,y) := x^2$$

$$= \int_{-2}^{2}\int_{x^2}^{4} x^2\,dy\,dx$$

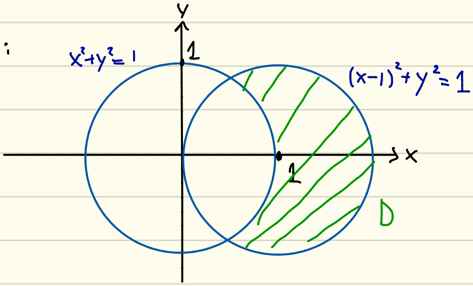
$$= \int_{-2}^{2}[x^2 y]_{x^2}^{4}\,dx = \int_{-2}^{2}(4x^2-x^4)\,dx = 2\int_0^2(4x^2-x^4)\,dx$$

$$= 2\left[\frac{4x^3}{3}-\frac{x^5}{5}\right]_0^2 = 2\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{128}{15}$$

Q3) Find the area of the region inside the circle $(x-1)^2+y^2=1$ and outside the circle $x^2+y^2=1$.

Sol) Step 1: Sketch the region D

Picture:

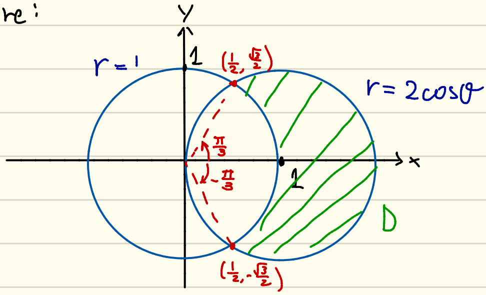


Step 2: describe D in polar coordinates:

Put $x = r\cos\theta$, $y = r\sin\theta$:
where $r>0$, $-\pi \leq \theta < \pi$

$$\begin{cases} x^2+y^2=1 \iff r=1 \\ x^2+y^2-2x=0 \iff r=2\cos\theta \end{cases}$$

Picture:



$$D = \left\{ (r,\theta) \in (0,+\infty)\times[-\pi,\pi) \;\middle|\; -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3},\ 1 \leq r \leq 2\cos\theta \right\}$$

Step 3: Evaluate the area using polar coordinate.

$$\text{Area} = \iint_D 1\,dA = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{r^2}{2}\right]_1^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} (4\cos^2\theta - 1)\,d\theta$$

$$= \int_0^{\frac{\pi}{3}} \left(4\cdot\left(\frac{1+\cos 2\theta}{2}\right) - 1\right) d\theta$$

$$= \int_0^{\frac{\pi}{3}} (1 + 2\cos 2\theta)\,d\theta$$

$$= \left[\theta + \sin 2\theta\right]_0^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$